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# An Approach for Combining the Ritz–Galerkin and Finite Element Methods

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## 1. INTRODUCTION

It is well known that elliptic boundary value problems can be solved by several numerical procedures, such as the Ritz–Galerkin method, the finite element method, the finite difference method, and the conservative difference scheme. However, it appears that there has been little or no work done on combining these methods. Of course, each of these methods has its advantages and shortcomings. The use of combined methods is particularly important in problems with complicated boundaries or boundary conditions, or in problems with solutions that are not smooth enough or have singularities, or in problems with unbounded solution domains. In such cases, a single method is often ineffective. In this paper we study a combined method which has been widely used.

In 1973, Strang and Fix [7, p. 135] mentioned the idea of combining the Ritz–Galerkin and finite element methods. In 1977, Zienkiewciz *et al.* [10] gave a systematic presentation of a combined method based on the boundary integral method and the finite element method, which cannot, however, be used for general nonhomogeneous equations.

In this paper, we introduce another combined method, which is nonconforming because the admissible functions are continuous only at the element nodes on the common boundary of both methods. However, it is usually the case that nonconforming effects are of little importance to the numerical solution obtained. This method has the advantages that it is valid for the general nonhomogeneous elliptic boundary value problem, and is reasonably simple to describe.



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## 2. The Combined Method

Consider the two-dimensional model problem,

$$-\frac{\partial}{\partial x}\left(\beta \frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(\beta \frac{\partial u}{\partial y}\right)=f, \qquad (x,y)\in S, \tag{2.1}$$

$$u=g, \qquad (x,y)\in \Gamma, \tag{2.2}$$

where $S$ is a convex polygon with boundary $\Gamma$, the functions $\beta$ and $f$ are sufficiently smooth, and $\beta=\beta(x,y)\geqslant\beta_0>0$, for some constant $\beta_0$. The model problem, (2.1) and (2.2), can be expressed in a weak form

$$a(u,v)=f(v), \qquad \forall v\in H_0^1(S), \tag{2.3}$$

where the true solution $u\in H_*^1(S)$.

$$a(u,v)=\int_S \beta(u_x v_x+u_y v_y), \tag{2.4}$$

$$f(v)=\int_S fv, \tag{2.5}$$

and the spaces are given by

$$H_*^1(S)=\{v, v_x, v_y\in L^2(S), v|_\Gamma=g\},$$
$$H_0^1(S)=\{v, v_x, v_y\in L^2(S), v|_\Gamma=0\}.$$

Let $S$ be divided by a circle $\Gamma_0$ into two subdomains: a circular domain $S_2$ contained in $S$ and another domain $S_1$ such that $S=S_1\cup S_2$. Two quite different methods, the linear finite element method and the Ritz–Galerkin method, are used on $S_1$ and $S_2$, respectively. Let $S_1$ be subdivided into small triangular elements $\Delta_i$ of maximum width $h$, and let the nodes of those elements adjacent to $S_2$ lie on $\Gamma_0$ (Fig. 1). The admissible functions for both methods are continuous only at the element nodes on $\Gamma_0$ so that this combination is nonconforming. This combination of the Ritz–Galerkin and finite element methods has been discussed in Li and Liang [5] for the case where common boundary $\Gamma_0$ is piecewise straight.

In order to simplify the method, isoparametric elements are not used; therefore the triangularized domain $\hat{S}_1^h$ only approximates $S_1$ (Fig. 1), i.e.,

$$\hat{S}_1^h=\bigcup_i \Delta_i\approx S_1. \tag{2.6}$$

The noncoincidence of $\hat{S}_1^h$ and $S_1$ may cause some difficulties; its effects on the numerical solution of the model problem will be studied in this paper.

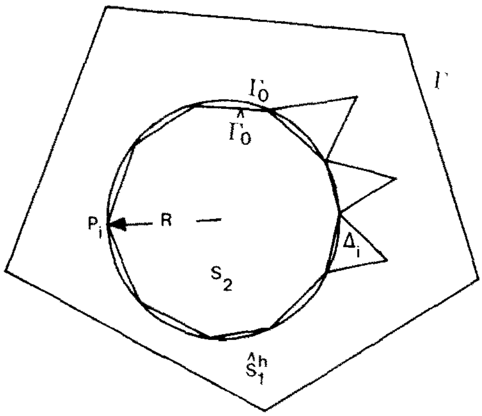


FIG. 1. The domain subdivisions in the combined method.

Let the admissible functions for the combined method be

$$\begin{aligned} v_h &= v_1, && (x, y)\hat{S}_1^h, \\ &= f_N = \sum_{l=0}^{N} d_l \phi_l, && (x, y) \in S_2, \end{aligned} \tag{2.7}$$

and

$$v_1(P_j) = f_N(P_j), \qquad P_j \in \Gamma_0. \tag{2.8}$$

where $v_1$ are piecewise linear interpolation functions on the triangularized domain $\hat{S}_1^h$, $\{\phi_l\}$ are complete, linearly independent basis functions, the $d_l$ are unknown coefficients, and $P_j$ are the element nodes on $\Gamma_0$. The space of functions $v_h$ satisfying (2.2) is denoted by $V_h^*$; the space of functions $v_h$ satisfying $v_h|_\Gamma = 0$ is denoted by $V_h^0$.

The combined method based on the Ritz–Galerkin method and the linear finite element method requires one to find an approximate solution $u_h^* \in V_h^*$ such that

$$a_h(u_h^*, v) = f(v), \qquad \forall v \in V_h^0, \tag{2.9}$$

where

$$a_h(u, v) = \sum_i \int_{\Delta_i} \beta(u_x v_x + u_y v_y) + \int_{S_2} \beta(u_x v_x + u_y v_y), \tag{2.10}$$

and

$$f(v) = \sum_i \int_{\Delta_i} fv + \int_{S_2} fv. \tag{2.11}$$

Note that the admissible functions $v_h$ are not continuous on $\Gamma_0$, except at the nodes $P_j$, hence

$$V_h^* \not\subset H_*^1(S) \qquad \text{and} \qquad V_h^0 \notin H_0^1(S). \tag{2.12}$$

In considering the above method, the following questions have, naturally, occurred to us:

Are the effects of (2.12) severe enough to prevent us from getting a good numerical solution?

Is the approximation (2.6) permitted?

What are the error bounds for the numerical solution of (2.9)?

We now investigate these questions.

The true solution $u$ on $S_2$ can be expanded as

$$u = \sum_{l=0}^{L} \left\{ \bar{a}_{l0} + \sum_{i=1}^{L} (\bar{a}_{li} \cos i\theta + \bar{b}_{li} \sin i\theta) \right\} T_l(r) + R_N, \qquad (r, \theta) \in S_2, \tag{2.13}$$

where the $T_l(r)$ are complete polynomials of order $l$, $\bar{a}_{li}$ and $\bar{b}_{li}$ are expansion coefficients, and $R_N$ is the remainder. Hence, it is reasonable to choose the following functions as the admissible functions on $S_2$:

$$f_N = \sum_{l=0}^{L} \left\{ a_{l0} + \sum_{i=1}^{L} (a_{li} \cos i\theta + b_{li} \sin i\theta) \right\} T_l(r), \tag{2.14}$$

where $a_{li}$ and $b_{li}$ are coefficients. To simplify the analyses, we assume that the subscripts $l$ and $i$ of the coefficients $a_{li}$ and $b_{li}$ are both bounded by some integer $L$.

We define a norm on $V_h^0$ as follows:

$$\|v\|_h = [\|v\|^2_{H^1(\hat{S}_1^h)} + \|v\|^2_{H^1(S_2)}]^{1/2},$$

where $\|\cdot\|_{H^1(S_2)}$ is the Sobolev norm [6]. Then $V_h^0$ is a Hilbert space, and the norm $\|v\|_h$ is also a measure of the mean value of $v$ and its generalized derivatives. We shall assess the error of the solution obtained by Eq. (2.9), in the norm $\|\cdot\|_h$.

We give a bound on the error of the solution in the following theorem; the proof of which is deferred to the next section.

THEOREM 1. *Let the admissible functions $f_N$ be given by* (2.14), *and $v_1$ be piecewise linear interpolation polynomials on $\hat{S}_1^h$ where the following inequality is assumed:*

$$h/\text{Min}\, h_i \leqslant K_0, \qquad \textit{for} \quad \{i \mid \Delta_i \textit{ is adjacent to } \Gamma_0\}. \tag{2.15}$$

*Here the $h_i$ are the maximum widths of the elements adjacent to $\Gamma_0$, $h$ is the maximum width over all elements, and $K_0$ is a bounded constant independent of $h$ and $h_i$.*

*Moreover, suppose that $a_h$ in (2.9) is uniformly $V_h^0$-elliptic, i.e., there exists a positive constant $\alpha$ independent of $h$ and $L$ such that*

$$\alpha \|v\|_h^2 \leqslant a_h(v, v), \qquad \forall v \in V_h^0. \tag{2.16}$$

*With these assumptions, the solution $u_h^*$ of (2.9) satisfies the following error bound:*

$$\begin{aligned} \|u - u_h^*\|_h \leqslant K_1 \{ & h\,|u|_{H^2(\hat{S}_1^h)} + \|R_N\|_{H^1(S_2)} + h^{1/2} [ |f|_{H^0(\Delta\hat{S}_1^h)} + |u|_{H^1(\Delta\hat{S}_1^h)} ] \\ & + (h^2L^2 + h^{3/2}) \left| \frac{\partial u}{\partial n} \right|_{H^0(\Gamma_0)} + h^{3/2}\,|u|_{H^2(\Gamma_0)} \\ & + \frac{1}{h^{1/2}} |R_N|_{H^0(\Gamma_0)} + h^{3/2}\,|R_N|_{H^2(\Gamma_0)} \}, \end{aligned} \tag{2.17}$$

*where $\Delta\hat{S}_1^h = S_1^h \wedge S_2$, $\partial u/\partial n$ is the normal derivative on $\Gamma_0$, and $K_1$ is a bounded constant independent of $h$ and $L$.*

Throughout this analysis $K_1$ represents a generic bounded constant with possibly different values in different contexts.

The first and second terms on the right side of (2.17) are the error bounds from the linear finite element method on $\hat{S}_1^h$ and the Ritz–Galerkin method on $S_2$, respectively. The third term is the error bound from the approximation (2.6), but it is at most $O(h^{3/2})$ because $\text{Meas}(\Delta\hat{S}_1^h) = o(h^2)$ and, in general, there exists the bound

$$\max_{(x, y) \in \Delta\hat{S}_1^h} \{ |f| + |u_x| + |u_y| \} \leqslant K_1.$$

The rest of the terms in (2.17) are from the nonconforming elements on $\Gamma_0$. Among these terms, the term $h^2L^2\,|\partial u/\partial n|_{H^0(\Gamma_0)}$ is the most important. In order to get an error bound of the form

$$\|u - u_h^*\|_h \leqslant K_1 h, \tag{2.18}$$

the inequality

$$L \leqslant K_1 h^{-1/2} \tag{2.19}$$

must hold. Consequently, an important problem for the combined method (2.9) is how to choose the integer $L$ for the Ritz–Galerkin method on $S_2$.

Inequality (2.19) usually holds because we are frequently able to obtain a valid expansion in (2.13). For example, suppose that the solution on $S_2$ has bounded partial derivatives of order $\mu$ ($\geqslant 3$). Then, we have the following bounds on the remainder [2, 3]:

$$\|R_N\|_{H^1(S_2)} \leqslant K_1 \frac{1}{L^{\mu-1}}, \tag{2.20}$$

$$|R_N|_{H^0(\Gamma_0)} \leqslant K_1 \frac{1}{L^{\mu}}, \tag{2.21}$$

and

$$|R_N|_{H^2(\Gamma_0)} = K_1 \frac{1}{L^{\mu-2}}. \tag{2.22}$$

Consequently, the error estimate (2.18) is obtained from Theorem 1 provided that the following optimal value for $L$ is chosen:

$$L = L_{\mathrm{opt}} = O(h^{-3/2\mu}) \qquad (\mu \geqslant 3). \tag{2.23}$$

In this case, inequality (2.19) naturally holds.

COROLLARY 1. *Suppose that all conditions in Theorem* 1 *hold, and $u$ on $S_2$ has bounded partial derivatives of order $\mu \geqslant 3$. Then, if we choose $L$ as in* (2.23), *the error estimate* (2.18) *holds.*

From (2.9), we obtain the algebraic system

$$A\mathbf{x} = \mathbf{b}, \tag{2.24}$$

where $\mathbf{x}$ is an unknown vector with the elements $(v_1)_{ij}$, $a_{li}$, and $b_{li}$, $b$ is a known vector, and the matrix $A$ is positive definitive, symmetric, and sparse. Hence, the numerical solutions of (2.24) or (2.9) are easily obtained.

The total number of coefficients $a_{li}$ and $b_{li}$ is $(2L+1)(L+1)$, which is at most $O(h^{-1})$ (see (2.23) or (2.19)). This is much less than $O(h^{-2})$, which is the number of the element nodes in the finite element method. Thus the use of the combined method can result in significant savings. Obviously, the larger the subdomain $S_2$, where $u$ has bounded partial derivatives of order $\mu \geqslant 3$, the less the cost of the calculation in (2.24).

Theorem 1 is still valid for arbitrary complete polynomials $\{T_l(r)\}$. In order to insure a stable solution to (2.9), we should require that the set $T_l(r)$ be orthogonal. For example, we may choose $T_l(r)$ to satisfy

$$\begin{aligned} \int_0^R r T_l(r)\, T_m(r)\, dr &= 0, \qquad l \neq m, \\ &= 1, \qquad l = m, \end{aligned}$$

with $R$ equal to the radius of $\Gamma_0$.

## 3. The Proof of Theorem 1

Let us first prove several lemmas.

Lemma 1. *Suppose that the uniformly $V_h^0$-elliptic inequality* (2.16) *holds; then*

$$\|u-u_h^*\|_h \leqslant K_1 \left\{ \lim_{v_h\in V_h^*} \|u-v_h\|_h \right.$$
$$+ \sup_{\omega_h\in V_h^0} \left| \int_{\Delta \hat{S}_1^h} [f\omega_h - \beta(u_x(\omega_h)_x + u_y(\omega_h)_y)] \right| / \|\omega_h\|_h$$
$$\left. + \left| \frac{\partial u}{\partial n} \right|_{H^0(\Gamma_0)} \sup_{\omega_h\in V_h^0} \left[ \int_{\Gamma_0} (\omega_h^+ - \omega_h^-)^2 \right]^{1/2} \Big/ \|\omega_h\|_h \right\}, \quad (3.1)$$

*where*

$$\omega_h^+ = \omega_h|_{S_2} = f_N, \qquad \omega_h^- = \omega_h|_{S_1} = v_1.$$

*Proof.* When (2.16) holds, the basic estimate for the error bound of nonconforming elements has been obtained by Strang and Fix [7, p. 178] and Ciarlet [1, p. 186]

$$\|u-u_h^*\|_h \leqslant K_1 \{ \inf_{v_h\in V_h^*} \|u-v_h\|_h + \sup_{\omega_h\in V_h^0} |f(\omega_h) - a_h(u,\omega_h)| / \|\omega_h\|_h \}. \quad (3.2)$$

By applying Green's theorem and (2.1), we see that

$$|f(\omega_h) - a_h(u,\omega_h)| = \int_{\Delta \hat{S}_1^h} \{ f\omega_h - \beta[u_x(\omega_h)_x + u_y(\omega_h)_y] \}$$
$$+ \int_{\partial S_1} \beta \frac{\partial u}{\partial n} \omega_h^- + \int_{\partial S_2} \beta \frac{\partial u}{\partial n} \omega_h^+, \quad (3.3)$$

where $\partial S_1$ and $\partial S_2$ are the boundaries of $S^1$ and $S_2$, respectively. Since the normal flow $\beta(\partial u/\partial n)$ on $\Gamma_0$ is continuous, it follows from Schwarz's inequality and $\omega_h^-|_\Gamma = 0$ that

$$\left| \int_{\partial S_1} \beta \frac{\partial u}{\partial n} \omega_h^- + \int_{\partial S_2} \beta \frac{\partial u}{\partial n} \omega_h^+ \right| \leqslant \left| \int_{\Gamma_0} \beta \frac{\partial u}{\partial n} (\omega_h^+ - \omega_h^-) \right|$$
$$\leqslant \left[ \int_{\Gamma_0} \left( \beta \frac{\partial u}{\partial n} \right)^2 \right]^{1/2} \left[ \int_{\Gamma_0} (\omega_h^+ - \omega_h^-)^2 \right]^{1/2}$$
$$\leqslant K_1 \left| \frac{\partial u}{\partial n} \right|_{H^0(\Gamma_0)} \left[ \int_{\Gamma_0} (\omega_h^+ - \omega_h^-)^2 \right]^{1/2}. \quad (3.4)$$

Consequently, inequality (3.1) is obtained from (3.2)–(3.4).

LEMMA 2. *Let $v_1$ in (2.7) be piecewise linear interpolation functions on $\hat{S}_1^h$, and $f_N$ be the functions (2.14), then*

$$\sup_{\omega_h \in V_h^0} \left[ \int_{\Gamma_0} (\omega_h^+ - \omega_h^-)^2 \right]^{1/2} \leqslant K_1 (h^2 L^2 + h^{3/2}) \|\omega_h\|_h. \tag{3.5}$$

*Proof.* We notice that the functions $\omega_h$ $(\in V_h^0)$ are continuous on the nodes $P_j \in \Gamma_0$, so that

$$\omega_h(P_j) = \omega_h^+(P_j) = \omega_h^-(P_j), \qquad \omega_h \in V_h^0.$$

A piecewise linear interpolation function with respect to $\theta$ is constructed as follows:

$$H(\theta) = \omega_h(R, \theta_j) + \frac{\omega_h(R, \theta_{j+1}) - \omega_h(R, \theta_j)}{\theta_{j+1} - \theta_j}, \tag{3.6}$$

where $(R, \theta_j)$ are the coordinates of the nodes $P_j$ on $\Gamma_0$ with $\theta_{j+1} > \theta_j$, and $R$ is the radius of the circle $\Gamma_0$. Then we have

$$\left[ \int_{\Gamma_0} (\omega_h^+ - \omega_h^-)^2 \right]^{1/2} \leqslant \left[ \int_{\Gamma_0} (\omega_h^+ - H(\theta))^2 \right]^{1/2} + \left[ \int_{\Gamma_0} (\omega_h^- - H(\theta))^2 \right]^{1/2}. \tag{3.7}$$

For the piecewise linear interpolation functions $H(\theta)$, we have the inequality (Ciarlet [1])

$$\int_{\Gamma_0} (\omega_h^+ - H(\theta))^2 \leqslant k_1 h^4 |\omega_h^+|^2_{H^2(\Gamma_0)}. \tag{3.8}$$

By applying $\omega_h^+ = f_N$, (2.14), and the orthogonality of trigonometric functions, after some calculations we find

$$\begin{aligned}
|\omega_h^+|^2_{H^2(\Gamma_0)} &= \int_{\Gamma_0} \left( \frac{\partial^2 \omega_h^+}{\partial \theta^2} \right)^2 \\
&= \pi \sum_{i=1}^{L} i^4 \left\{ \left[ \sum_{l=1}^{L} a_{li} T_l(R) \right]^2 + \left[ \sum_{l=1}^{L} b_{li} T_l(R) \right]^2 \right\} R \\
&\leqslant L^4 \left[ 2\pi R \left( \sum_{l=0}^{L} a_{l0} \right)^2 + \pi \sum_{i=1}^{L} \left\{ \left[ \sum_{l=0}^{L} a_{li} T_l(R) \right]^2 \right. \right. \\
&\qquad \left. \left. + \left[ \sum_{l=0}^{L} b_{li} T_l(R) \right]^2 \right\} R \right] \\
&= L^4 \int_{\Gamma_0} (\omega_h^+)^2 \leqslant K_1 L^4 \|\omega_h\|_h^2,
\end{aligned} \tag{3.9}$$

where in the last step we have used the imbedding theorem of Sobolev [6].

Then, since

$$\begin{aligned}\omega_h &= a_i + b_i x + c_i y \\ &= a_i + b_i r\cos\theta + c_i r\sin\theta. \qquad (x, y) \in \Delta_i,\end{aligned}$$

where $a_i$, $b_i$, and $c_i$ are constants, we find that

$$\begin{aligned}\int_{\Gamma_0} |\omega_h - H(\theta)|^2 &\leqslant K_1 \sum_i h_i^4 \int_{\theta_i}^{\theta_{i+1}} \left(\frac{\partial^2 \omega_h}{\partial\theta^2}\right)^2 R\, d\theta \\ &= K_1 R^2 \sum_i h_i^4 \int_{\theta_i}^{\theta_{i+1}} (b_i \cos\theta + c_i \sin\theta)^2 R\, d\theta \\ &\leqslant 2K_1 R^2 \sum_i h_i^4 R(\theta_{i+1} - \theta_i)(b_i^2 + c_i^2) \\ &= 2K_1 R^2 \sum_i h_i^4 \frac{R(\theta_{i+1} - \theta_i)}{\text{Meas}(\Delta_i)} \int_{\Delta_i} \left[\left(\frac{\partial\omega_h}{\partial x}\right)^2 + \left(\frac{\partial\omega_h}{\partial y}\right)^2\right] \\ &\leqslant K_2 h^3 \|\omega_h\|_h^2, \end{aligned} \tag{3.10}$$

where $K_2$ is a bounded constant independent of $h$, and in the last step we have used the Sobolev imbedding theorem and the inequality

$$R(\theta_{i+1} - \theta_i)/\text{Meas}(\Delta_i) \leqslant K_1/h_i,$$

for regular triangular elements $\Delta_i$.

Inequality (3.5) is then obtained from (3.7)–(3.10).

For estimating the bounds of $\inf_{v_h \in V_h} \|u - v_h\|_h$ in Lemma 1, let us construct an auxiliary function $\bar{w}_h \in V_h^*$ in the following way:

$$\begin{aligned}\bar{w}_h &= \bar{u}_h, && (x, y) \in \hat{S}_1^h, \\ &= \bar{f}_N = \sum_{l=0}^{L} \left\{\bar{a}_{l0} + \sum_{i=1}^{L} [\bar{a}_{li}\cos i\theta + \bar{b}_{li}\sin i\theta]\, T_l(r)\right\}, && (x, y) \in S_2,\end{aligned} \tag{3.11}$$

where $\bar{f}_N$ is the approximate expansion of $u$ (see (2.13)), and $\bar{u}_h$ is also a piecewise linear interpolation function on $\hat{S}_1^h$ but its values on the element nodes $P_i$ of $\hat{S}_1^h$ are given by:

$$\begin{aligned}\bar{u}_h(P_i) &= u(P_i), && P_i \notin \Gamma_0, \\ &= \bar{f}_N(P_i), && P_i \in \Gamma_0.\end{aligned} \tag{3.12}$$

Next, let $u_h$ be a piecewise linear interpolation function of $u$ on $\hat{S}_1^h$, with $u_h(P_i) = u(P_i)$ for all element nodes $P_i$ on $\hat{S}_1^h$. We have

LEMMA 3. *Suppose* (2.15) *holds; then*

$$\|u_h - \bar{u}_h\|_{H^1(\hat{S}_1^h)} \leqslant K_1 \frac{1}{h^{1/2}} |u_h - \bar{u}_h|_{H^0(\Gamma_0)}. \tag{3.13}$$

*Proof.* We can, as in [4], obtain the following inequality from (2.15):

$$\|u_h - \bar{u}_h\|^2_{H^1(\hat{S}_1^h)} \leqslant K_1 \frac{1}{h} |u_h - \bar{u}_h|^2_{H^0(\hat{\Gamma}_0)}, \tag{3.14}$$

where $\hat{\Gamma}_0$ is the interior boundary of $\hat{S}_1^h$. Note that the norm on the right of (3.14) is on $\hat{\Gamma}_0$, while the norm on the right of (3.13) is on $\Gamma_0$.

Let $P_h = u_h - \bar{u}_h$. Since the distance between $\hat{\Gamma}_0$ and $\Gamma_0$ is, at most, $o(h^2)$, we have

$$\begin{aligned} |P_h|^2_{H^0(\hat{\Gamma}_0)} - |P_h|^2_{H^0(\Gamma_0)} &\leqslant K_1 h^2 \left| \frac{\partial}{\partial r} (P_h)^2 \right|_{H^0(\Gamma_0)} \\ &= 2K_1 h^2 \left| P_h \cdot \frac{\partial}{\partial n} P_h \right|_{H^0(\Gamma_0)} \\ &\leqslant 2K_1 h^2 |P_h|_{H^0(\Gamma_0)} \left| \frac{\partial}{\partial n} P_h \right|_{H^0(\Gamma_0)}. \end{aligned} \tag{3.15}$$

For the piecewise linear function $P_h$ we have, as in the proof of (3.10), that

$$\left| \frac{\partial P_h}{\partial n} \right|_{H^0(\Gamma_0)} \leqslant K_1 \frac{1}{h^{1/2}} \|P_h\|_{H^1(\hat{S}_1^h)}. \tag{3.16}$$

By combining (3.14)–(3.16), the following inequality is obtained:

$$x^2 \leqslant K_1(2bx + c), \tag{3.17}$$

with $x = \|P_h\|_{H^1(\hat{S}_1^h)}$, $b = h^{1/2} |P_h|_{H^0(\Gamma_0)}$, and $c = |P_h|^2_{H^0(\Gamma_0)}/h$. Hence, we obtain the following bound on $x$:

$$x \leqslant K_1[b + (b^2 + c)^{1/2}],$$

i.e.,

$$\begin{aligned} \|P_h\|_{H^1(\hat{S}_1^h)} &\leqslant K_1 \left[ h^{1/2} + \left( h + \frac{1}{h} \right)^{1/2} \right] |P_h|_{H^0(\Gamma_0)} \\ &\leqslant K_2 h^{-1/2} |P_h|_{H^0(\Gamma_0)}, \end{aligned} \tag{3.18}$$

with $K_2$ a bounded constant independent of $h$. This gives inequality (3.13) by noting $P_h = u_h - \bar{u}_h$.

Let us now prove Theorem 1 along the lines of Lemma 1.

The bound on the third term on the right side of (3.1) has been estimated in Lemma 2. As for a bound on the second term, we see from the Schwarz's inequality and Strang and Fix [7, p. 169] that

$$\left| \int_{\Delta \hat{S}_1^h} \{f\omega_h - \beta[u_x(\omega_h)_x + u_y(\omega_h)_y]\} \right|$$
$$\leqslant K_1 [|f|_{H^0(\Delta \hat{S}_1^h)} + |u|_{H^1(\Delta \hat{S}_1^h)}] \|\omega_h\|_{H^1(\Delta \hat{S}_1^h)}$$
$$\leqslant K_2 h^{1/2} [|f|_{H^0(\Delta \hat{S}_1^h)} + |u|_{H^1(\Delta \hat{S}_1^h)}] \|\omega_h\|_h, \tag{3.19}$$

where $K_2$ is a constant independent of $h$.

The bound for the first term is rather complex. Since $\bar{\omega}_h$, defined by (3.11), belongs to $V_h^*$, we have

$$\inf_{v_h \in V_h} \|u - v_h\|_h \leqslant \|u - \bar{\omega}_h\|_h$$
$$\leqslant \|u - \bar{u}_h\|_{H_1(\hat{S}_1^h)} + \|R_N\|_{H^1(S_2)}, \tag{3.20}$$

with the remainder $R_N = u - \bar{f}_N$.

We see from Lemma 3 and the piecewise linear interpolation function $u_h$ of $u$ that

$$\|u - \bar{u}_h\|_{H^1(\hat{S}_1^h)} \leqslant \|u - u_h\|_{H^1(\hat{S}_1^h)} + \|u_h - \bar{u}_h\|_{H^1(\hat{S}_1^h)}$$
$$\leqslant h\,|u|_{H^2(\hat{S}_1^h)} + \frac{1}{h^{1/2}}\,|u_h - \bar{u}_h|_{H^0(\Gamma_0)}. \tag{3.21}$$

We note that the piecewise linear interpolation function $\bar{u}_h$ satisfies (3.12); then

$$|u_h - \bar{u}_h|_{H^0(\Gamma_0)} \leqslant |u - u_h|_{H^0(\Gamma_0)} + |\bar{f}_N - \bar{u}_h|_{H^0(\Gamma_0)} + |u - \bar{f}_N|_{H^0(\Gamma_0)}$$
$$\leqslant K_1 [h^2\,|u|_{H^2(\Gamma_0)} + h^2\,|\bar{f}_N|_{H^2(\Gamma_0)}] + |R_N|_{H^0(\Gamma_0)}$$
$$\leqslant K_1 [2h^2\,|u|_{H^2(\Gamma_0)} + h^2\,|R_N|_{H^2(\Gamma_0)}] + |R_N|_{H^0(\Gamma_0)}, \tag{3.22}$$

where in the last step we have used the inequality

$$|\bar{f}_N|_{H^2(\Gamma_0)} = |u - R_N|_{H^2(\Gamma_0)}$$
$$\leqslant |u|_{H^2(\Gamma_0)} + |R_N|_{H^2(\Gamma_0)}.$$

Hence, the bound on the first term in (3.1) is given (from (3.20)–(3.22)) by

$$\inf_{v_h \in V_h} \|u - v_h\|_h \leqslant K_1 \Big\{ h\,|u|_{H^2(\hat{S}_1^h)} + \|R_N\|_{H^1(S_2)} + h^{3/2}\,|u|_{H^2(\Gamma_0)}$$
$$+ \frac{1}{h^{1/2}}\,|R_N|_{H^0(\Gamma_0)} + h^{3/2}\,|R_N|_{H^2(\Gamma_0)} \Big\}. \tag{3.23}$$

Finally, inequality (2.17) is obtained from (3.1), (3.5), (3.19), and (3.23). This completes the proof of Theorem 1.

The uniformly $V_h^0$-elliptic inequality (2.16) can be proved (as in [5]), so the conditions in Theorem 1 are satisfied.

*Remark.*, Suppose the common boundary $\Gamma_0$ is a curve instead of a circle. After some calculation, it can be shown that

$$|\omega_h^\cdot|^2_{H^2(\Gamma_0)} \leqslant K_1 L^{2m},$$

with $m$ a real number $\geqslant 2$. Hence, Theorem 1 still holds even for a curved common boundary $\Gamma_0$, but with the general bound $K_1(h^2L^m + h^{3/2}) |\partial u/\partial n|_{H^0(\Gamma_0)}$ instead of $K_1(h^2L^2 + h^{3/2}) |\partial u/\partial n|_{H^0(\Gamma_0)}$ in (2.17).

## 4. The Simple Case

In this section, we consider the simple case where the functions in (2.1) satisfy

$$\beta \equiv \text{const} \qquad \text{and} \qquad f \equiv 0, \quad (x, y) \in S_2. \tag{4.1}$$

Here, the essential assumption is that $\beta \equiv$ const on $S_2$. In fact, for a nonhomogeneous equation $\beta \Delta u = f$, with a constant $\beta$ on $S_2$, a particular solution $u^*$ can be frequently found such that $\beta \Delta u^* = f$ on $S_2$. Hence, if we define a new variable $v = u - u^*$, Eq. (2.1) reduces to a homogeneous equation $\beta \Delta v = 0$ on $S_2$(i.e., $f \equiv 0$ on $S_2$).

The assumptions (4.1) mean that Laplace's equation holds on $S_2$

$$\Delta u = 0, \qquad (x, y) \in S_2. \tag{4.2}$$

Its solutions can be expanded as

$$u(r, \theta) = a_0 + \sum_{l=1}^{L} (\bar{a}_l \cos l\theta + \bar{b}_l \sin l\theta) \left(\frac{r}{R}\right)^l + R_l, \tag{4.3}$$

where the expansion coefficients are

$$\bar{a}_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(R, \theta)\, d\theta, \qquad \bar{a}_l = \frac{1}{\pi} \int_{\pi}^{\pi} u(R, \theta) \cos l\theta\, d\theta, \qquad l \geqslant 1,$$

$$\bar{b}_l = \frac{1}{\pi} \int_{-\pi}^{\pi} u(R, \theta) \sin l\theta\, d\theta, \qquad l \geqslant 1, \tag{4.4}$$

and the remainder term is

$$R_L = \sum_{l=L+1}^{\infty} (\bar{a}_l \cos l\theta + \bar{b}_l \sin l\theta) \left(\frac{r}{R}\right)^l. \tag{4.5}$$

Hence, the following admissible functions are better than those of (2.14):

$$f_N = f_L = a_0 + \sum_{l=1}^{L} (a_l \cos l\theta + b_l \sin l\theta)\left(\frac{r}{R}\right)^l, \qquad (x, y) \in S_2, \tag{4.6}$$

with the unknown coefficients $a_l$ and $b_l$, the total number of which is only $(2L + 1)$; this is less than $(2L + 1)(L + 1)$, the total number of unknown coefficients in (2.14).

Because the admissible functions (4.6) satisfy Laplace's equation (4.2), we find from Green's theorem that for $u \in V_h^*$ and $v \in V_h^0$,

$$\int_{S_2} \beta[u_x v_x + u_y v_y] = \beta \int_{\Gamma_0} \frac{\partial u}{\partial n} v. \tag{4.7}$$

Consequently, the combined method (2.9) can be written in the simple form:

$$\bar{a}_h(u_h^*, v) = \bar{f}(v), \tag{4.8}$$

where

$$\bar{a}_h(u, v) = \sum_i \int_{\Delta_i} \beta(u_x v_x + u_y v_y) + \beta \int_{\Gamma_0} \frac{\partial u^+}{\partial n} v^+, \tag{4.9}$$

and

$$\bar{f}(v) = \sum_i \int_{\Delta_i} fv \qquad \text{and} \qquad u^+ = f_N = u|_{S_2}.$$

The method (4.8) is concerned only with $\hat{S}_1^h$ and $\Gamma_0$, where it is somewhat like the coupling method of Zienkewciz *et al.* [10]. However, the latter method cannot be used for the general equation (2.1).

Using the orthogonality property of trigonometric functions and the fact that $\Gamma_0$ is a circle, we obtain

$$\beta \int_{\Gamma_0} \frac{\partial u_h^+}{\partial n} v^+ = \beta\pi \sum_{l=1}^{L} l(a_l \tilde{a}_l + b_l \tilde{b}_l), \tag{4.10}$$

for $u_h^+ = f_N$ and

$$v = \tilde{a}_0 + \sum_{l=1}^{L} (\tilde{a}_l \cos l\theta + \tilde{b}_l \sin l\theta)\left(\frac{r}{R}\right)^l. \tag{4.11}$$

Hence, the algebraic system (2.24) is immediately obtained from (4.8), where the admissible functions satisfy the constraint conditions (2.8), i.e.,

$$v_1(R, \theta_j) = a_0 + \sum_{l=1}^{L} (a_l \cos l\theta_j + b_l \sin l\theta_j). \tag{4.12}$$

In a similar manner, there is also a simplification of the original form (2.9) of the combined method with the admissible functions (2.14).

THEOREM 2. *Assume* (4.1) *holds, and suppose that all the conditions in Theorem* 1 *hold, except for* (2.14) *which we replace with the condition* (4.6). *Then, the solution* $u_h^*$ *of* (4.8) *has the error bound*

$$\begin{aligned}\|u-u_h^*\|_h \leqslant K_1 \Big\{ & h\,|u|_{H^2(\hat{S}_1^h)} + \left[|R_L|_{H^0(\Gamma_0)} \left|\frac{\partial R_L}{\partial n}\right|_{H^0(\Gamma_0)}\right]^{1/2} \\ & + h^{1/2}\big[|f|_{H^0(\Delta\hat{S}_1^h)} + |u|_{H^1(\Delta\hat{S}_1^h)}\big] \\ & + (h^2L^2+h^{3/2}) \left|\frac{\partial u}{\partial n}\right|_{H^0(\Gamma_0)} + h^{3/2}\,|u|_{H^2(\Gamma_0)} \\ & + \frac{1}{h^{1/2}}|R_L|_{H^0(\Gamma_0)} + h^{3/2}\,|R_L|_{H^2(\Gamma_0)}\Big\}. \end{aligned} \tag{4.13}$$

*Proof.* Theorem 1 clearly holds for the particular case in this section. Then, from a comparison with (2.17), we see that Theorem 2 holds provided that

$$\|R_L\|_{H^1(S_2)} \leqslant (1+R)\left[|R_L|_{H^0(\Gamma_0)} \left|\frac{\partial R_L}{\partial n}\right|_{H^0(\Gamma_0)}\right]^{1/2}. \tag{4.14}$$

We now prove (4.14). With the orthogonality of trigonometric functions, we have the inequality for the norms $|\cdot|_{H^0(S_2)}$ and $|\cdot|^2_{H^1(S_2)}$ of $R_l$ given by (4.5),

$$\begin{aligned}|R_L|^2_{H^0(S_2)} &= R^2 \sum_{l=L+1}^{\infty} \frac{\pi(\bar{a}_l^2+\bar{b}_l^2)}{2l+2} \\ &\leqslant R^2 \sum_{l=L+1}^{\infty} \pi(\bar{a}_l^2+\bar{b}_l^2)\, l = R^2\,|R_L|^2_{H^1(S_2)}. \end{aligned} \tag{4.15}$$

Since the remainder $R_L$ also satisfies Laplace's equation, then it follows from Green's theorem that

$$\begin{aligned}|R_L|^2_{H^1(S_2)} &= \int_{\Gamma_0} R_L \frac{\partial R_L}{\partial n} \\ &\leqslant |R_L|_{H^0(\Gamma_0)} \left|\frac{\partial R_L}{\partial n}\right|_{H^0(\Gamma_0)} \end{aligned} \tag{4.16}$$

Hence, we see from (4.15) and (4.16) that

$$\|R_L\|^2_{H^1(S_2)} = |R_L|^2_{H^0(S_2)} + |R_L|^2_{H^1(S_2)}$$
$$\leqslant (1+R^2)|R_L|^2_{H^1(S_2)}$$
$$\leqslant (1+R^2)|R_L|_{H^0(\Gamma_0)}\left|\frac{\partial R_L}{\partial n}\right|_{H^0(\Gamma_0)}. \qquad (4.17)$$

This completes the proof of Theorem 2.

Suppose that $u$ has bounded partial derivatives of order $\mu$ ($\geqslant 3$) on $\Gamma_0$. Then, we have the bounds (2.21), (2.22) and

$$\left|\frac{\partial R_L}{\partial n}\right|_{H^0(\Gamma_0)} \leqslant K_1 \frac{1}{L^{\mu-1}}. \qquad (4.18)$$

This leads to

COROLLARY 2. *Suppose that all conditions in Theorem 2 hold, and $u$ has bounded partial derivatives of $\mu$ ($\geqslant 3$) on $\Gamma_0$. Then, if the integer $L$ is chosen as in* (2.23), *the error estimate* (2.18) *still holds.*

In Theorem 2 and Corollary 2, the bounds of the norms are concerned only with $\hat{S}_1^h$ and $\Gamma_0$. This is an advantage for the analyses of singularity problems where there is only a little change in the form of admissible functions in $S_2$ (see next section).

## 5. SINGULARITY PROBLEMS

The combined method ((2.9) or (4.8)) is very efficient for solving the singularity problems. Here we consider two examples.

### 5.1. *The Crack Problem*

For simplicity, Laplace's equation

$$\Delta u = 0, \qquad (x, y) \in S, \qquad (5.1)$$

is considered; the boundary conditions are

$$u|_{\Gamma_1} = g_1, \qquad (5.2)$$

and

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma_2} = g_2. \qquad (5.3)$$

Let $\Gamma = \Gamma_1 \cup \Gamma_2$ and let a crack lie on the $x$ axis (Fig. 2). Suppose that the boundary condition on the crack is given by

$$u|_{\Gamma} = 0 \qquad (y = 0, x \geqslant 0). \qquad (5.4)$$

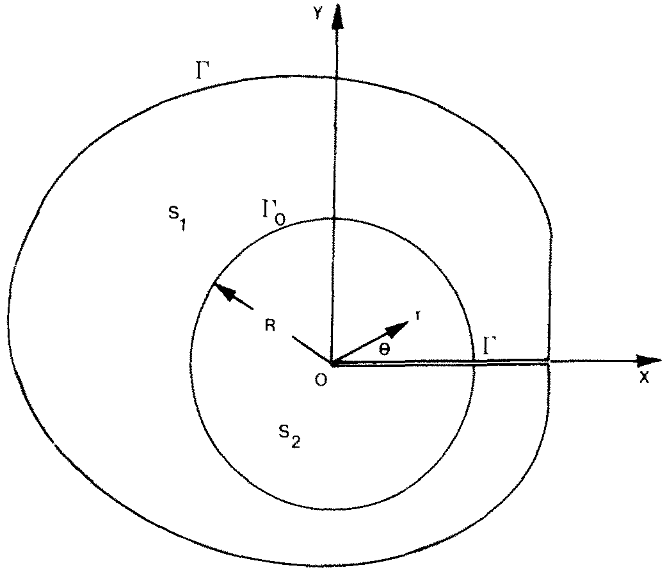


FIG. 2. The crack problem.

There exists a singularity at the origin which is contained by $S_2$. Using the method of separation of variables, the solution $u|_{S_2}$ can be expanded as

$$u = \sum_{n=1}^{L} \bar{a}_n \left(\frac{r}{R}\right)^{n+(1/2)} \sin\left(n + \frac{1}{2}\right)\theta + R_L, \tag{5.5}$$

with the coefficients

$$\bar{a}_n = \int_0^{2\pi} u(R, \theta) \sin(n + \tfrac{1}{2})\,\theta\, d\theta,$$

and the remainder

$$R_L = \sum_{n=L+1}^{\infty} \bar{a}_n \left(\frac{r}{R}\right)^{n+(1/2)} \sin\left(n + \frac{1}{2}\right)\theta, \qquad (r, \theta) \in S_2. \tag{5.6}$$

Hence, the admissible functions should be chosen as

$$f_N = f_L = \sum_{n=1}^{L} a_n \left(\frac{r}{R}\right)^{n+(1/2)} \sin\left(n + \frac{1}{2}\right)\theta, \qquad (r, \theta) \in S_2, \tag{5.7}$$

with unknown coefficients $a_n$. Then the solution of the crack problem (5.1)–(5.4) is easily obtained from (4.8) with the admissible functions (5.7) instead of (4.6).

### 5.2. *Motz's Problem*

Motz's problem is another typical singularity problem which involves Laplace's equation (5.1) on a rectangular domain $S$ $(-1 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1)$, with the following boundary conditions [8, 9]:

$$\left.\frac{\partial u}{\partial n}\right|_{y=1} = \left.\frac{\partial u}{\partial n}\right|_{x=-1} = \left.\frac{\partial u}{\partial n}\right|_{y=0 \wedge x>0} = 0, \tag{5.8}$$

$$u|_{y=0 \wedge x<0} = 0, \tag{5.9}$$

and

$$u|_{x=1} = 500. \tag{5.10}$$

There is a singularity at the origin, which is contained by $S_2$ (Fig. 3). The solution on $S_2$ is

$$u = \sum_{l=0}^{L} \bar{a}_l \left(\frac{r}{R}\right)^{l+(1/2)} \cos\left(l + \frac{1}{2}\right)\theta + R_L, \qquad (r, \theta) \in S_2, \tag{5.11}$$

with the coefficients $\bar{a}_l = \int_0^\pi u(R, \theta) \cos(l + \frac{1}{2})\,\theta$, and the remainder

$$R_L = \sum_{l=L+1}^{\infty} \bar{a}_l \left(\frac{r}{R}\right)^{l+(1/2)} \cos\left(l + \frac{1}{2}\right)\theta.$$

Then, we take the admissible functions

$$f_N = f_l = \sum_{l=0}^{L} a_l \left(\frac{r}{R}\right)^{l+(1/2)} \cos\left(l + \frac{1}{2}\right)\theta, \qquad (r, \theta) \in S_2. \tag{5.12}$$

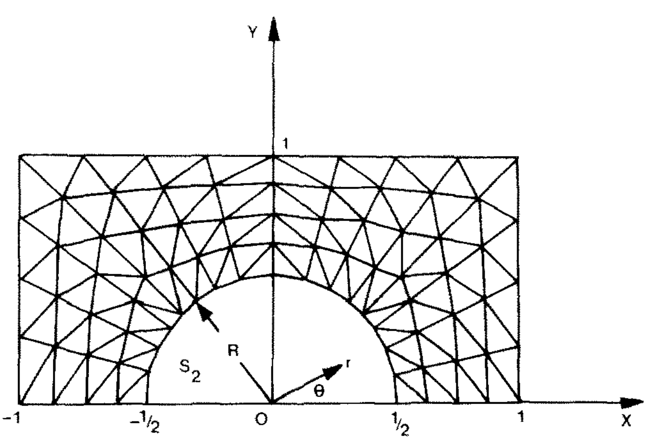


FIG. 3. Motz's problem.

TABLE I

Numerical Solutions of Motz's Problem

| $(x_i, y_i)$ | $(-\frac{2}{7}, \frac{2}{7})$ | $(0, \frac{2}{7})$ | $(\frac{2}{7}, \frac{2}{7})$ | $(-\frac{1}{28}, \frac{1}{28})$ | $(0, \frac{1}{28})$ | $(\frac{1}{28}, \frac{1}{28})$ | $(\frac{1}{28}, 0)$ |
|---|---|---|---|---|---|---|---|
| Combined method | 78.4732 | 141.133 | 243.567 | 33.5478 | 53.1192 | 83.5686 | 76.3152 |
| Thatcher's method | 78.24 | 140.9 | 243.3 | 33.37 | 52.89 | 83.20 | 76.01 |
| W and P's method [9] | 78.56 | 141.6 | 243.8 | 33.59 | 53.19 | 83.67 | 76.41 |

with the unknown coefficients $a_l$. The numerical solution of Motz's problem is then obtained from (4.8) with the admissible functions (5.12).

## 6. Numerical Examples

Two numerical examples are given in this section.

### 6.1. *Motz's Problem*

The division of $S_1$, shown in Fig. 3, is similar to the infinite grid refinement method of Thatcher [8], but there is a nonuniformity in the distribution of the element nodes on the circle $\Gamma_0$ of radius $R = \frac{1}{2}$. The numerical solution calculated by the combined method with $L = 4$ is listed in Tables I and II. For comparison, the numerical results of Thatcher [8] are also listed in Tables I and II, from which we see that the solutions calculated by these different methods are in approximate agreement.

TABLE II

The Coefficients $a_l$ for Motz's Problem

| $l$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Combined method | 400.665 | 87.7679 | 17.6683 | −9.66311 | 1.79988 |
| Thatcher's method | 400.8 | 88.0 | 17.3 | — | — |
| Symm's method | 401.2 | 87.2 | — | | |

The combined method (4.8) is relatively simple to use but, in Thatcher's method, an eigenvalue problem must first be solved for the case shown in Fig. 3.

We notice that only five unknown coefficients, $a_0$–$a_4$, need to be calculated so that a saving in the cost of the calculation is possible in the combined method.

### 6.2. *A Common Problem*

Consider Lapalce's equation (5.1) on a semicircular domain $S$ $(0 < r < 1,$ $0 < \theta < \pi)$ with the boundary conditions (Fig. 4)

$$\left.\frac{\partial u}{\partial n}\right|_{\theta=0,\pi} = 0 \qquad \text{and} \qquad u|_{r=R} = e^{\theta} \qquad (0 \leqslant \theta \leqslant \pi). \tag{6.1}$$

The true solution is

$$u(r,\theta) = \bar{a}_0 + \sum_{l=1}^{L} \bar{a}_l r^l \cos l\theta \qquad (0 \leqslant r \leqslant 1), \tag{6.2}$$

with the coefficients

$$\bar{a}_0 = \frac{1}{\pi}(e^{\pi} - 1), \tag{6.3}$$

$$\bar{a}_l = \frac{1}{(1+l^2)\pi}[(-1)^l e^{\pi} - 1] \qquad (l \geqslant 1). \tag{6.4}$$

Hence, the admissible functions on $S_2$ are chosen to be

$$f_N = a_0 + \sum_{l=1}^{L} a_l r^l \cos l\theta.$$

The linear finite element method is used in $S_1$ with the triangulation shown in Fig. 4.

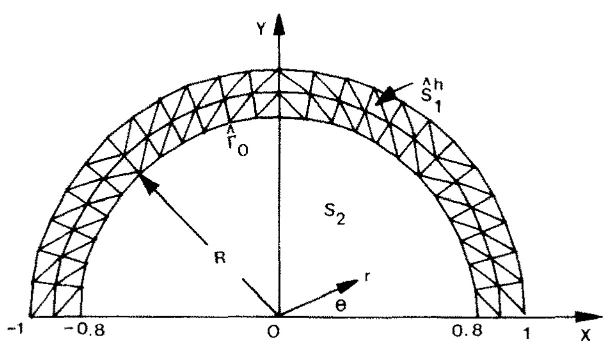


FIG. 4. The domain subdivision for the common problem.

TABLE III

The Numerical Solutions of Eqs. (5.1) and (6.1)

| $(x_i, y_i)$ | (0.8, 0) | (0.4, 0) | (0, 0) | (−0.4, 0) | (−0.8, 0) | (0, 0.4) | (0, 0.8) | Maximal Error |
|---|---|---|---|---|---|---|---|---|
| Combined method | 2.116 | 4.347 | 7.058 | 10.723 | 16.738 | 6.624 | 5.516 | 0.144 |
| Finite element method | 2.111 | 4.342 | 7.058 | 10.732 | 16.769 | 6.623 | 5.511 | 0.175 |
| Exact solution | 2.117 | 4.343 | 7.048 | 10.700 | 16.594 | 6.616 | 5.509 | — |

For comparison, the solution is also calculated by the linear finite element method on the whole solution domain $S$, where the triangulation on $S_1$ is the same as in Fig. 4, and the triangulation on $S_2$ is similar to that on $S_1$.

The solutions obtained by the combined method with $L = 16$ and the finite element method are listed in Table III and IV. It is shown in Table III that both solutions are in approximate agreement. However, there are 42 and 226 unknown quantities to be calculated in the combined method and the finite element method, respectively. Therefore, the combined method is beneficial even for a common boundary value problem.

## Concluding Remarks

From the above analysis and numerical examples, it is clear that the combined method described in this paper should be used for singularity problems. Moreover, we also recommend that the combined method be used for solving the general boundary value problem if there exists a large subdomain where the solution is sufficiently smooth.

TABLE IV

The Calculated Coefficients $a_l$ for Eqs. (5.1) and (6.1)

| $l$ | 0 | 1 | 2 | 3 | 4 | 6 | 8 | 16 |
|---|---|---|---|---|---|---|---|---|
| Combined method | 7.058 | −7.702 | 2.837 | −1.563 | 0.859 | 0.422 | 0.278 | 0.179 |
| Exact solution | 7.048 | −7.684 | 2.819 | −1.637 | 0.829 | 0.381 | 0.227 | 0.055 |

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